Experiment 2.5: Calculation of magnetostatic fields and energies

Introduction

Numerical integration is often used in cases where analytical solutions are not available. One particular example is the calculation of the magnetic fields from, and the energies of, magnetic structures. In certain cases the fields and energies can be calculated exactly. This is true for systems with simple geometries and uniform magnetisation; however, in general it is necessary to use numerical integration. In this experiment you will develop code to calculate the magnetic field and energies of bodies using a finite difference representation.

Objectives

- To develop code for the numerical calculation of magnetic fields and energies.
- To apply the model to calculations of magnetic energies for simple systems in order to test the code against analytical results.
- To compare the energies of different magnetic states of a magnetic platelet.
- to calculate the magnetic energies of two interacting platelets depending on their magnetic configuration.

Magnetostatic field and the scalar potential

The magnetostatic field is calculated using Maxwell's equations for magnetostatics,

$$\nabla \cdot \mathbf{B} = 0, \tag{1}$$

and

$$\nabla \times \mathbf{H} = 0. \tag{2}$$

Equations 1 and 2 must be solved subject to the boundary conditions (under the assumption of no current) that the normal component of **B** and tangential component of **H** are continuous. Because the curl of **H** is zero we can write the field as the gradient of a scalar potential, i.e.

$$\mathbf{H} = -\nabla\phi. \tag{3}$$

Substitution into Eq. 1 and using the definition $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$ gives

$$\nabla^2\phi = 4\pi\nabla \cdot \mathbf{M}. \tag{4}$$

In terms of the scalar potential the boundary conditions relate the internal and external values at the interface as follows;

$$\phi_{int} = \phi_{ext} \tag{5}$$

and

$$\frac{\partial \phi}{\partial n}\Big|_{ext} - \frac{\partial \phi}{\partial n}\Big|_{int} = -4\pi \mathbf{M}. \tag{6}$$

The formal solution to Eq. 4 subject to these boundary conditions leads to a field given by

$$\begin{aligned} \mathbf{H}(\mathbf{r}) &= \int_{S'} \frac{(\mathbf{r}-\mathbf{r}')\mathbf{M}(\mathbf{r}').\hat{n}}{|(\mathbf{r}-\mathbf{r}')|^3} dS' \\ &- \int_{V'} \frac{(\mathbf{r}-\mathbf{r}')\nabla\cdot\mathbf{M}(\mathbf{r}')}{|(\mathbf{r}-\mathbf{r}')|^3} dV', \end{aligned} \tag{7}$$

where the integrals are over the surface S' and volume V' of the volume of the magnetic material. Equation 7 gives the magnetic field at any point $\mathbf{r}$, which can be inside or outside the magnetic material.
The total energy of the system is

$$E_{tot} = \frac{1}{2}\int \mathbf{H}\cdot\mathbf{M} dV' \tag{8}$$

Numerical integration

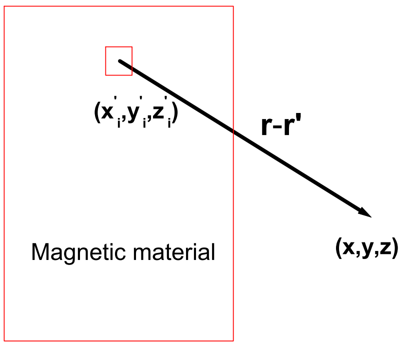

Figure 1: Schematic of the numerical integration process

In order to carry out the numerical integration we here use a simple approach, outlined in fig 1. Essentially the technique divides the integration space into small cells of volume $\Delta V' = \Delta x'\Delta y'\Delta z'$, situated at discrete points x'_i, y'_i, z'_i, which represent the coordinates of the centre of the volume element. The function to be integrated is assumed to be piecewise constant over this volume. The integral of some function $f(x, y, z)$ is then approximated by

$$\int_V f(x,y,z)dV = \Delta V \sum_i f(x_i, y_i, z_i), \tag{9}$$

where the summation runs over all the discrete cells in the integration volume. Note that rather than working with continuous variables we are now using discrete variables, which can be dealt with numerically. For example, the field calculation involves $\mathbf{r} - \mathbf{r}'$. Remembering that $\mathbf{r}'$ is the integration variable over the volume of the magnetic material, in discrete form $\mathbf{r} - \mathbf{r}'$ is written

$$\mathbf{r} - \mathbf{r}' = \hat{\mathbf{x}}(x - x_i) + \hat{\mathbf{y}}(y - y_i) + \hat{\mathbf{z}}(z - z_i), \tag{10}$$

where $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ are the unit vectors, and $\mathbf{r} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z$ is the point at which the field is to be calculated.
Also, you will need the unit vector $\hat{\mathbf{n}}$, which is normal to the surface. For cuboids, assuming that the sides are parallel to the (x, y, z) axes, this is easy to calculate; $\hat{\mathbf{n}} = \pm\hat{\mathbf{z}}$ (the sign dependent on whether $\hat{\mathbf{n}}$ points long the positive or negative direction) assuming that the z-direction is normal to the surface, and $\hat{\mathbf{n}} = \pm\hat{\mathbf{x}}$ and $\hat{\mathbf{n}} = \pm\hat{\mathbf{y}}$ for the other 2 surfaces.

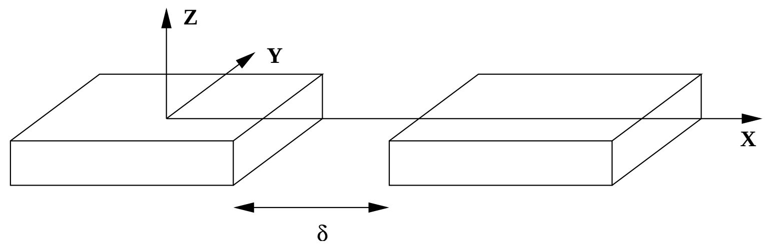

Figure 2: Geometry for the calculation of the energy of 2 interacting platelets separated by a distance δ.

Experiment

1. Write code to calculate the magnetic field at any point (inside or outside) a volume of magnetic material of uniform magnetisation $\mathbf{M}$, using Eq. 7. Note that in the case of a uniform magnetisation the volume integral is zero and the calculation reduces to the evaluation of an integral over the surface of the magnetised body. Plot the results using arrows to represent the magnitude and direction of the magnetic field at any point in space.

2. Write code to calculate the energy of the magnetised material using Eq. 8.

3. Test your code by calculating the energy of a platelet of dimensions 10x10x1μm, with the magnetisation perpendicular to the face of the platelet. Approximating this geometry to an infinite platelet, the magnetic field is uniform, of magnitude $4\pi M$ and directed antiparallel to the magnetisation. Using Eq. 8 this gives for the energy of the system.

$$\begin{aligned} E &= \frac{1}{2} 4\pi M^2 \int_V dV' \\ &= 2\pi M^2 V, \end{aligned} \tag{11}$$

where V is the volume of the platelet. Compare the energy of the system with magnetisation perpendicular to the plane with the case when the magnetisation lies in the plane. Which is the preferred direction for the magnetisation?

4. Calculate the energy a system of two platelets of dimensions 10x10x1μm^3 with their long axes oriented in the x and y directions as shown in Fig 2. The platelets are separated by a distance $\delta = 1\mu m$ and the magnetisation directions pointing along the z-axis. Calculate the energy when the directions of magnetisation are parallel, ie with the magnetisation of both platelets along the $+z$ direction, and the anti-parallel direction where the magnetisation of one platelet is reversed. Which is the preferred state? How does the interaction energy change with δ?