Experiment 2.3: Non-Linear Behaviour and Chaos: the Duffing Oscillator

Introduction

The theory of chaos studies complex nonlinear dynamic systems, which are non-constant and non-periodic. In these systems small changes of the initial conditions can cause very large fluctuations, so that it becomes practically impossible to predict the exact state of the system after some time has passed. The attention must then be shifted to the analysis of the overall behaviour of these systems, looking for global characteristics which can characterise them. It turns out that such overall behaviour can be captured e.g. by considering the system evolution in a Poincarré plot, where chaotic behaviour is associated to the appearance of the so-called “strange attractors”.
A change in the environment (e.g. a strong applied periodic force), can induce in some systems extreme sensitivity on initial conditions; in this experiment we shall study the behaviour of a particle moving under the influence of a potential field and driven by an external force. The equation of motion is

$$\frac{d^2y_1}{dt^2} = -Ay_1(t)^3 + By_1(t) - C\frac{dy_1}{dt} + D\sin(\omega t). \tag{1}$$

This equation is non-linear, and includes a damping term proportional to the particle velocity. From Eq. (1) we see that the external potential is given by

$$V = Ay_1^4/4 - By_1^2/2 \tag{2}$$

– a double-well potential – where the quartic term confines the system and provides the restoring force, while the quadratic term pushes the particle away from the origin.
The system can become chaotic when driven by a periodic force ($D \neq 0$).
To study the motion of our particle we must solve Eq. (1). Differential equations can be solved computationally using different algorithms. In this experiment we shall use the (2,2) Runge-Kutta method.

Objectives

- to solve computationally a system of differential equations
- to analyse computationally how the behaviour of the solutions change as parameters and initial conditions are varied
- to study the onset and signature of chaos in a particular system (Duffing oscillator)

Experiment

The potential field Eq. (2) confining our system presents a double-well structure characterised by an unstable equilibrium point at $y_1 = 0$.

1. Implement the Runge-Kutta method and test it against a second order differential equation of which you know the exact solution (e.g. $d^2y_1/dt^2 = -\omega y_1$)
2. Study the oscillator in the non-driven case ($D = 0$). What happens if the oscillator has/has not sufficient initial energy to move across the unstable equilibrium point? Plot trajectory and phase-space diagram in both cases and comment on them.
3. Weakly driven case: consider small values for D and set the particle at the unstable equilibrium point as initial condition. How do the trajectory and phase-space diagram change? Tuning the parameters, you should be able to observe a simple limit cycle, i.e. a motion which is asymptotically periodic with the frequency of the driving force. Consider the Poincarré section for this case and comment on it.
4. Period doubling: increase the strength of the driving force. Plot the Poincarré section and the trajectory corresponding to the ensamble of snapshot points you have used to plot the Poincarré section. How do trajectory and Poincarré plot change? Consider if there are asymptotic accumulation points in it and if their number changes. If necessary, discharge from your graph the points corresponding to an initial transient. Period doubling is one of the possible route of a system toward a chaotic behaviour.
5. Strongly driven case: increase the strength of the driving force further.How do trajectory and phase diagram change? Plot the Poincarré section. If the behaviour is chaotic, the plot will show a fractal pattern which is named “strange attractor”(see fig. 1).
6. Varying the control parameters C, D and ω changes the shape of such an attractor. Explore different combinations.

(2,2) Runge-Kutta method

Eq. (1) is a second order differential equation and can be rewritten as a system of two coupled differential equations

$$\frac{dy_1}{dt} = f_1(t, y_1, y_2) = y_2 \tag{3}$$

$$\frac{dy_2}{dt} = f_2(t, y_1, y_2) = -Ay_1(t)^3 + By_1(t) - Cy_2 + D\sin(\omega t). \tag{4}$$

The time interval of interest is then discretised in n steps of width h. This parameter will determine the accuracy of the solution, the error being of order $o(h^3)$.
The solutions to Eq. (3) and (4) at the $(i+1)$-th step are given by

$$y_{1,i+1} = y_{1,i} + \frac{K_{11} + K_2}{2} \tag{5}$$

$$y_{2,i+1} = y_{2,i} + \frac{K_{22} + K_1}{2} \tag{6}$$

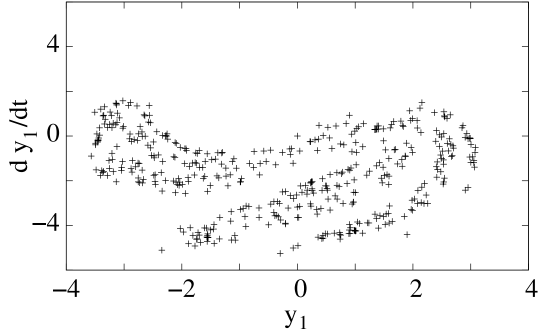

Figure 1: Poincarré plot showing an example of strange attractor

with

$$
\begin{aligned}
K_{11} &= hf_1(t_i, y_{1,i}, y_{2,i}) = K_{21} & (7)\\
K_{22} &= hf_2(t_i, y_{1,i}, y_{2,i}) = K_{12} & (8)\\
K_1 &= hf_2(t_i + h, y_{1,i} + K_{21}, y_{2,i} + K_{22}) & (9)\\
K_2 &= hf_1(t_i + h, y_{1,i} + K_{11}, y_{2,i} + K_{12}). & (10)
\end{aligned}
$$

To initialise the calculation it is sufficient to provide the values of y_1 and y_2 at the chosen initial time t_0.
To test the method it is convenient to consider, as a first step, a system of differential equations whose solution is known. In any case it is important to check if the value chosen for h is appropriate for the precision required by the problem.

Glossary

Phase-space plot: plot of velocity against displacement. If velocity and displacement uniquely define the motion, different trajectories will not cross in this plot.
Poincarré plot: it consists of snapshots of the phase-space taken at regular time intervals, such that $t\omega = 2n\pi$, $n = 0, 1, 2...$, where ω is the frequency of the driving force. Regular oscillations of the system should then correspond to a single point.